Towards three-loop QCD corrections to the time-like splitting functions

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in collaboration with

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Three-loop time-like $q \to g$ splitting function

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Why three-loop?

Because since 1980 two-loop splitting functions were calculated by various methods probably as more times as any other expression.

Space-like

- axial gauge (PV prescription) Curci, Furmanski, Petronzio '80; Ellis, Vogelsang '98
- Feynman gauge (x-space) Floratos, Kounnas, Lacaze '81
- axial gauge (ML prescription) Bassetto, Heinrich, Kunszt, Vogelsang '98
- Feynman gauge (Mellin space) Moch, Vermaseren '99
- axial gauge (NPV prescription) OG, Jadach, Skrzypek, Kusina '14 (A.Kusina's talk)

Time-like

- axial gauge (PV prescription) Furmanski, Petronzio '80
- Feynman gauge (x-space) Floratos, Kounnas, Lacaze '81
- analytic continuation Stratmann, Vogelsang '96; Blumlein, Ravindran, van Neerven '00; Moch, Vogt '07
- Feynman gauge (Mellin space) Mitov, Moch '06

Three-loop time-like $q \to g$ splitting function

Why time-like?

Because space-like three-loop splitting functions are already calculated. Moch, Vermaseren, Vogt '04
(*Nuclear Physics B* selected top paper in 2004)

But can't one use some trick to derive them from space-like results?

Examples of tricks (analytic continuation): Drell, Levy, Yan '70; Gribov, Lipatov '72

Sure!

- NNLO non-singlet Mitov, Moch, Vogt '06
- NNLO singlet ($q \to q$ and $g \to g$) Moch, Vogt '07
- NNLO singlet ($q \to g$ and $g \to q$) Almasy, Moch, Vogt '11

Three-loop time-like q $\to$ g splitting function should be calculated directly.

1. Mass factorization (algebraic) relations

The unpolarized differential cross-section for "$\gamma^* \to$ partons" decay

$$\frac{1}{\sigma_{\rm tot}} \frac{d^2\sigma^H}{dx\, d\cos\theta} = \frac{3}{8}(1+\cos^2\theta)\,\mathcal{F}_T(x) + \frac{3}{4}\sin^2\theta\,\mathcal{F}_L(x) + \frac{3}{4}\cos\theta\,\mathcal{F}_A(x)$$

The mass factorization relations are Vermaseren, Vogt, Moch '05

$$\mathcal{F}_T^{(1)} = -\frac{2}{\epsilon} P^{(0)}_{\rm gq} + c^{(1)}_{T,\rm g} + \epsilon\, a^{(1)}_{T,\rm g} + \epsilon^2 b^{(1)}_{T,\rm g}$$

$$\begin{aligned}
\mathcal{F}_T^{(2)} = {} & \frac{1}{\epsilon^2}\left\{ P^{(0)}_{\rm gq}\left(P^{(0)}_{\rm qq} + P^{(0)}_{\rm gg} + \beta_0\right)\right\} + \frac{1}{\epsilon}\left\{ P^{(1)}_{\rm gq} + 2\,c^{(1)}_{T,\rm q} P^{(0)}_{\rm gq} + c^{(1)}_{T,\rm g} P^{(0)}_{\rm gg}\right\} \\
& + c^{(2)}_{T,\rm g} - 2\,a^{(1)}_{T,\rm q} P^{(0)}_{\rm gq} - a^{(1)}_{T,\rm g} P^{(0)}_{\rm gg} + \epsilon\left\{ a^{(2)}_{T,\rm g} - 2\,b^{(1)}_{T,\rm q} P^{(0)}_{\rm gq} - b^{(1)}_{T,\rm g} P^{(0)}_{\rm gg}\right\}
\end{aligned}$$

$$\begin{aligned}
\mathcal{F}_T^{(3)} = {} & -\frac{1}{6\epsilon^3}\left\{ P^{(0)}_{\rm gi} P^{(0)}_{\rm ij} P^{(0)}_{\rm jg} + 3\beta_0\, P^{(0)}_{\rm gi} P^{(0)}_{\rm ig} + 2\,\beta_0^2\, P^{(0)}_{\rm gg}\right\} \\
& + \frac{1}{6\epsilon^2}\left\{ 2P^{(0)}_{\rm gi} P^{(1)}_{\rm ig} + P^{(1)}_{\rm gi} P^{(0)}_{\rm ig} + 2\beta_0\, P^{(1)}_{\rm gg} + 2\beta_1\, P^{(0)}_{\rm gg} + 3P^{(0)}_{\rm gi}\left(P^{(0)}_{\rm ij} + \beta_0\delta_{\rm ij}\right) c^{(1)}_{\phi,\rm j}\right\} \\
& - \frac{1}{6\epsilon}\left\{ 2\,P^{(2)}_{\rm gg} + 3\,P^{(1)}_{\rm gi} c^{(1)}_{\phi,\rm i} + 6\,P^{(0)}_{\rm gi} c^{(2)}_{\phi,\rm i} - 3P^{(0)}_{\rm gi}\left(P^{(0)}_{\rm ij} + \beta_0\delta_{\rm ij}\right) a^{(1)}_{\phi,\rm j}\right\} \\
& + c^{(3)}_{\phi,\rm g} - \frac{1}{2} P^{(1)}_{\rm gi} a^{(1)}_{\phi,\rm i} - P^{(0)}_{\rm gi} a^{(2)}_{\phi,\rm i} + \frac{1}{2} P^{(0)}_{\rm gi}\left(P^{(0)}_{\rm ij} + \beta_0\delta_{\rm ij}\right) b^{(1)}_{\phi,\rm j} + O(\epsilon)
\end{aligned}$$

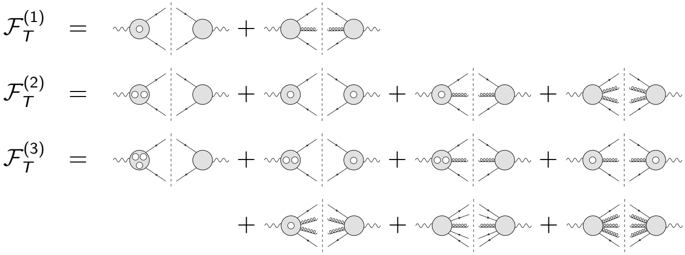
2. Fragmentation Functions

$$\mathcal{F}_T(x,\epsilon) \quad = \quad \frac{2}{2-d}\left(\frac{k_0\cdot q}{q^2}\,W_\mu{}^\mu + \frac{k_0^\mu k_0^\nu}{k_0\cdot q}\,W_{\mu\nu}\right)$$

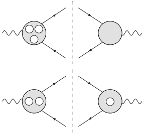
- The hadronic tensor is defined as

$$W_{\mu\nu}(x,\epsilon) = \frac{x^{d-3}}{4\pi}\int \mathrm{dPS(n)}\; M_\mu(n)\; M_\nu(n)$$

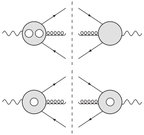
- $\mathrm{dPS(n)}$ is n-particle real phase-space
- amplitude $M^\mu(n)$ describes process "$\gamma^*(q) \to g(k_0) + n$ partons"



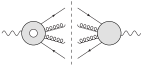
3. Final Considerations for NNLO corrections



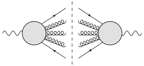
Pure-virtual contributions contain overall $\delta(1-x)$ factor.
We do not consider such contributions.



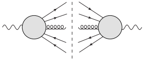
Can be extracted from Garland, Gehrmann, Glover, Koukoutsakis, Remiddi '01
Calculated by Duhr, Gehrman, Jaquier 1411.3587 [hep-ph]



One-loop helicity amplitudes by Bern, Dixon, Kosower '97
Final-state integration is of NLO complexity — simple.

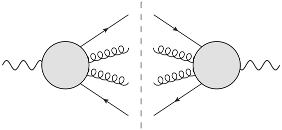


Contribution is known from analytical continuation by Almasy, Moch, Vogt '11

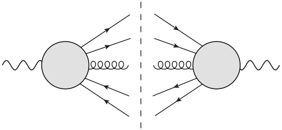


Unknown!

Final-state integration



$\sim$ 3-loop



$\sim$ 4-loop

The main challenge of the calculation is *n*-particle final-state integration:

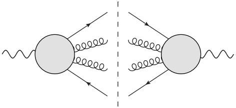
$$\int \mathrm{dPS(n)} = \int \prod_{i=0}^{n} \mathrm{d}^m k_i \delta^+(k_i^2)\, \delta\left(x - \frac{2k_0 \cdot q}{q^2}\right)\, \delta(q - \sum_{j=0}^{n} k_j)$$

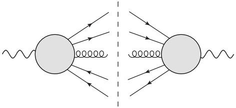
Since recently such integrals can be found with Integration-By-Parts method.

Software we consider:

- `LiteRed` Lee '13
- `Reduze` von Manteuffel, Studerus '12

Preparation





`QGRAF`	▶ 8 amplitudes	▶ 48 amplitudes
`FORM`		
▶ trace of gamma matrices		
▶ index contraction		
▶ color traces		
▶ partial fractioning		
`Mathematica`	▶ 499 integrals	▶ 55 614 integrals
▶ analyze symmetries		
▶ split by topologies		
`LiteRed`		
▶ find IBP reduction rules	▶ $\sim$10 h	▶ $\sim$1000 h (multi-thread)
▶ find masters	▶ 9 masters	▶ $\sim$80 masters

System of Differential Equations for Masters at NLO

$$
\begin{pmatrix}
\frac{(2\epsilon-1)(2x-1)}{(1-x)x} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{3\epsilon-2}{(1-x)x} & -\frac{3\epsilon-1}{x} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{(2\epsilon-1)(3\epsilon-1)}{\epsilon(x-1)x} & \frac{2\epsilon}{1-x} & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{(2\epsilon-1)(3\epsilon-2)(3\epsilon-1)}{2\epsilon(x-1)x^2} & \frac{(2\epsilon-1)(3\epsilon-1)\left(x^2-10x+1\right)}{2(1-x)x^2(x+1)} & 0 & \frac{2\epsilon\left(x^2-3x-2\right)}{(1-x)x(x+1)} & \frac{2\epsilon(6\epsilon-1)}{(1-x)x} & 0 & 0 & 0 & 0 \\
0 & \frac{(2\epsilon-1)(3\epsilon-1)}{\epsilon(x-1)x} & 0 & \frac{2}{x-1} & \frac{6\epsilon-1}{1-x} & 0 & 0 & 0 & 0 \\
\frac{4(2\epsilon-1)(3\epsilon-2)(3\epsilon-1)}{\epsilon^2(1-x)x^3(x+1)} & \frac{4(2\epsilon-1)(3\epsilon-1)\left(x^2-x+1\right)}{\epsilon(x-1)x^3(x+1)^2} & 0 & \frac{4\left(x^2+1\right)}{(x-1)x^2(x+1)^2} & \frac{2(6\epsilon-1)}{(1-x)x^2(x+1)} & \frac{(2\epsilon+1)(2x+1)}{-x(x+1)} & 0 & 0 & 0 \\
\frac{2(2\epsilon-1)(3\epsilon-2)(3\epsilon-1)}{\epsilon^2(x-1)^2x^2} & -\frac{(2\epsilon-1)(3\epsilon-1)}{\epsilon(x-1)x^2} & \frac{2\epsilon}{(x-1)x} & 0 & 0 & 0 & \frac{4\epsilon+1}{-x} & 0 & 0 \\
\frac{2(1-2\epsilon)(3\epsilon-2)(3\epsilon-1)}{\epsilon^2(x-1)^2x^3} & \frac{2(2\epsilon-1)(3\epsilon-1)(3x-1)}{\epsilon(x-1)^3x^3} & \frac{4\epsilon}{(1-x)^2x} & \frac{4\left(x^2+1\right)}{(x-1)^3x^2} & \frac{2(6\epsilon-1)(x+1)}{(1-x)^3x^2} & 0 & 0 & \frac{(2\epsilon+1)(2x-1)}{(1-x)x} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{(2\epsilon+1)(2x-1)}{(1-x)x}
\end{pmatrix}
$$

Following properties ensure that solution can be found in terms of Harmonic Polylogarythms Remiddi, Vermaseren '99:

- Alphabet (letters): $\{x, 1-x, 1+x\}$
- Main diagonal contains only $\left\{\frac{1}{x}, \frac{1}{1-x}, \frac{1}{1+x}\right\}$ terms
- Non-triangular terms vanish in $\epsilon \to 0$ limit

Solutions for Masters at NLO

Masters are found in terms of HPLs
as ϵ-series to any order.

$$M_2(x) = M_2^{(0)} + \epsilon\, M_2^{(1)} + \epsilon^2\, M_2^{(2)} + \dots$$

$$M_2^{(0)} = x^{1-3\epsilon}\left(C_2^{(0)} - 2\ln x\, C_1^{(0)}\right)$$

$$M_2^{(1)} = x^{1-3\epsilon}\left(C_2^{(1)} - 2\ln x\, C_1^{(1)} + (3\mathrm{H}_0 - 4\mathrm{H}_2 - 2\mathrm{H}_{0,0})\, C_1^{(0)}\right)$$

$$M_2^{(2)} = x^{1-3\epsilon}\Big(C_2^{(2)} - 2\ln x\, C_1^{(2)} + (3\mathrm{H}_0 - 4\mathrm{H}_2 - 2\mathrm{H}_{0,0})\, C_1^{(1)}$$
$$+ (6\mathrm{H}_2 - 4\mathrm{H}_3 + 3\mathrm{H}_{0,0} - 4\mathrm{H}_{2,0} - 8\mathrm{H}_{2,1} - 2\mathrm{H}_{0,0,0})\, C_1^{(0)}\Big)$$

Actual challange is to find integration constants $C_n^{(k)}$!

To be continued...

Summary

Done:

- **Preparation steps** to start integration
  $\rightarrow$ code in `QGRAF`, `FORM`, `Mathematica`
- **Simplify** integrals
  $\rightarrow$ various symmetries and partial fractioning
- **IBP identities** for NNLO case
  $\rightarrow$ reduces 55 614 integrals to just $\sim$ 80 masters
  $\rightarrow$ $\sim$ 1000h and still running
  (`CC1` cluster at IFJ PAN & `Phenod` cluster at DESY)
- **Solution for masters**
  $\rightarrow$ from differential equations to any order in ϵ-parameter

In progress:

- Implementation of the boundary conditions finder
- Three-loop time-like $q \rightarrow g$ splitting function