Fuchsia and Master Integrals for Energy-Energy Correlations at NLO in QCD

Oleksandr Gituliar
with Sven Moch

II. Institut für Theoretische Physik
Universität Hamburg

Matter to the Deepest 2017
Podlesice (Poland)

Agenda

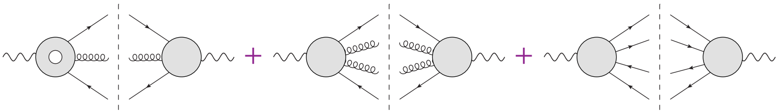
Energy-Energy Correlations in QCD

Definition

$$\Sigma(\xi) = \sum_{a,b} \int \mathrm{dPS}\, \frac{E_a\, E_b}{Q^2}\, \sigma\big(e^+ + e^- \to a + b + X\big)\, \delta(\xi - \cos\theta_{ab})$$

- infrared-safe quantity (UV and IR poles cancel)

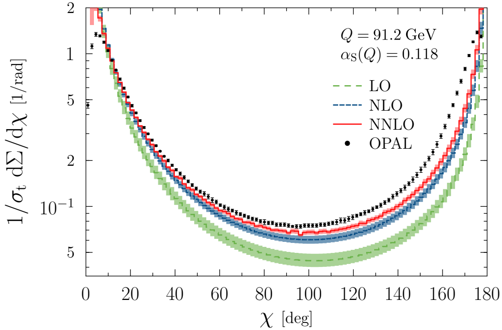
NLO contributions (virtual + real)



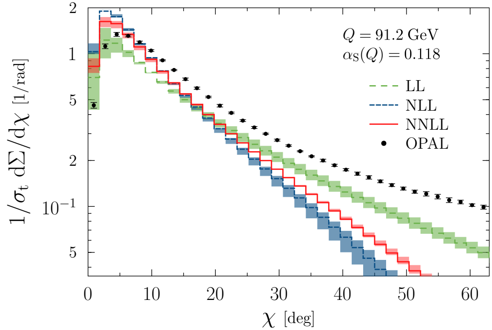
- correlated particles — a, b (e.g., qg, $\bar{q}g$, $q\bar{q}$, gg)
- angle variable — θ_{ab} or ξ
- energies — E_a, E_b
- phase-space — dPS

Status

Fixed Order



Resummation



plots from 1708.04093

Numerical Results

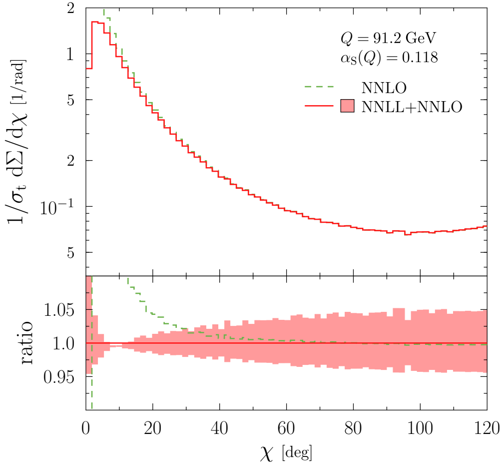
- 2010 — NNLL Becher, Neuberg ’10 (see also de Florian, Grazzini ’05)
- 2016 — NNLO Del Duca et al.
- 2017 — NNLL + NNLO Tulipánt, Kardos, Somogyi

Analytical Results

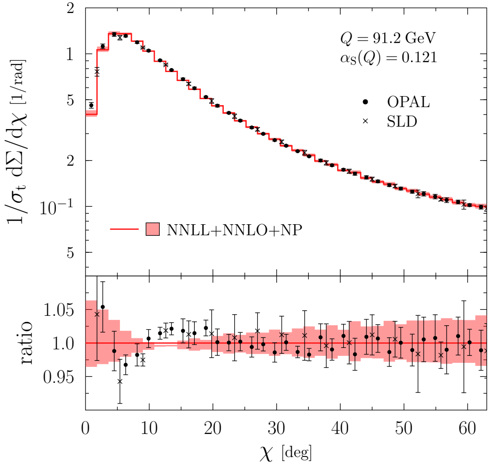
- 1978 — LO Basham, et al. ’78
- 2017 — NLO?

Phenomenology

Fixed Order + Resummation



+ Data Fit



plots from 1708.04093

Data Fit

- non-perturbative (NP) terms Dokshitzer, Marchesini, Webber '99

I.0 Preparation

Definition

$$\Sigma(\xi) = \sum_{a,b} \int \mathrm{dPS}\, \frac{E_a\, E_b}{Q^2}\, \sigma\big(e^+ + e^- \to a + b + X\big)\, \delta\big(\xi - \cos\theta_{ab}\big)$$

Process $\quad e^+ + e^- \to q(p_1) + \bar{q}(p_2) + g(p_3) + g(p_4), \qquad p_i^2 = 0$

Parametrization

- more convenient *angle variable*

$$z = \frac{1-\xi}{2} = \frac{p_1 \cdot p_3}{2\, q\cdot p_1\, q\cdot p_3}$$

- phase-space

$$\mathrm{dPS} = \mathrm{d}^m p_1\, \delta(p_1^2)\, \mathrm{d}^m p_2\, \delta(p_2^2)\, \mathrm{d}^m p^3\, \delta(p_3^2)\, \delta(q - p_1 - p_2 - p_3)^2$$

Linearization

$$\delta\left(z - \frac{p_1 \cdot p_3}{2\, q\cdot p_1\, q\cdot p_3}\right) \to \int_0^1 \mathrm{d}x\, x\, q\cdot p_3\; \delta\big(x - 2\, q\cdot p_1\big)\, \delta\big(x\, z\, q\cdot p_3 - p_1 \cdot p_3\big)$$

I.1 IBP Reduction

Cross-section $\sigma\left(e^+ + e^- \to q + \bar{q} + g + g\right)$

- 377 KB text file by `FormCalc` Hahn ’98 ’17

Cutkosky rules for Cut Propagators

$$\delta\left(p_1^2\right)\delta\left(p_2^2\right)\delta\left(p_3^2\right)\delta\left((q-p_1-p_2-p_3)^2\right)\delta\left(x-2\,q\cdot p_1\right)\delta\left(x\,z\,q\cdot p_3-p_1\cdot p_3\right)$$

Reduction

- do partial fraction decomposition
- identify topologies
- run `LiteRed` Lee ’12
  - 11 masters $F(x,z,\epsilon)$ (max 3×3 coupled subsystem)

$$F_1=\{\}\quad F_2=\{2\}\quad F_3=\{2,2\}\quad F_4=\{2,6\}\quad F_5=\{1,2\}$$
$$F_6=\{5\}\quad F_7=\{1,4,5\}\quad F_8=\{2,3,4\}\quad F_9=\{2,5\}\quad F_{10}=\{3,5\}\quad F_{11}=\{2,4,5\}$$

  - denominators

$$D_1=(k_2+k_3)^2\qquad D_2=(q-k_2)^2\qquad D_3=(q-k_1-k_2)^2$$
$$D_4=(q-k_1-k_3)^2\quad D_5=(q-k_2-k_3)^2\quad D_6=\left(k_1^2\right)_{\mathrm{x}}$$

Feynman Integrals Calculus

$$\int \underbrace{\mathrm{d}^d l_1 \ldots \mathrm{d}^d l_n}_{\text{loops}} \underbrace{\mathrm{d}^d p_1 \delta(p_1^2) \ldots \mathrm{d}^d p_m \delta(p_m^2)}_{\text{legs}} \frac{\delta(..)\ldots}{D_1^{n_1} \ldots D_k^{n_k}}$$

Numerical methods:

- Sector Decomposition/Mellin-Barnes
- Subtraction Schemes

Analytical methods

- Feynman/Schwinger parametrization
- Integration-By-Parts reduction Chetyrkin, Tkachov '81
  - Laporta algorithm Laporta '00: `AIR`, `FIRE`, `Kira`, `Reduze2`
  - Symbolic reduction: `LiteRed` Lee '12
  - private implementations
- Method of Differential Equations Kotikov '91, Remiddi '97
  - Epsilon form Henn '13
  - Lee algorithm Lee '14: `Fuchsia` OG, Magerya '16, `Epsilon` Prausa '17
  - `Canonica` Meyer '17

Method of Differential Equations

Construct System of ODE

- from definition
  - Holonomic Functions
  - Hypergeometric Function
- from IBP rules
  - `AIR`, `FIRE`, `Kira`, `LiteRed`, `Reduze2`

Solve System of ODE

- epsilon form Henn '13
  - Lee algorithm: `Fuchsia`, `epsilon`
  - Meyer algorithm: `Canonica`
  - works well with *hyperlogarithms*
  - helpless with *elliptic functions*
    - elliptic polylogarithms? Weinzierl et al. '16
- by other means

Find Constants of Integration

- no systematic approach
- usually the hardest step

I.2 Differential Equations for $F(x,z,\epsilon)$

ODE Definition

$$\frac{\mathrm{d}F(x,z,\epsilon)}{\mathrm{d}z} = A(x,z,\epsilon)\,F(x,z,\epsilon)$$

- z as free variable
- IBP rules (step II)
- 11×11 matrix
- alphabet

$$z, \quad z-1, \quad z-\frac{1}{x}, \quad z-\frac{1}{x\,(x-2)}$$

Epsilon form

- multivariate problem
- `Fuchsia` (Lee algorithm)

$$\frac{\mathrm{d}\hat{F}(x,z,\epsilon)}{\mathrm{d}z} = \epsilon\,B(x,z)\,\hat{F}(x,z,\epsilon), \quad \text{where} \quad F(x,z,\epsilon) = \boxed{T(x,z,\epsilon)}\,\hat{F}(x,z,\epsilon)$$

Solutions for $\hat{F}(x,z,\epsilon)$

- fully automated
- any ϵ-order

I.2 Example: $F_4(x,z,\epsilon)$

$$
\begin{aligned}
F_4(x,z,\epsilon) = {} & \frac{1}{15\,\epsilon^2}\Big(C_3^0(x) - 2C_2^0(x)\Big) + \frac{1}{30\,\epsilon}\bigg(\Big(15C_1^0(x) + 4C_2^0(x) - 6xC_3^0(x) - 2xC_4^0(x)\Big)\,\mathrm{H}_0(z) \\
& + \Big(\frac{15}{1-x}C_1^0(x) + 2C_3^0(x) - 2xC_4^0(x)\Big)\,\mathrm{H}_1(z) \\
& + \Big(\frac{15(x-2)}{1-x}C_1^0(x) + 20C_2^0(x) + 2(3x-7)C_3^0(x) + 4xC_4^0(x)\Big)\,\mathrm{H}_{1/x}(z) \\
& - \frac{15(1-2z)}{xz(1-z)}C_1^0(x) + \frac{4(13xz-1)}{xz}C_2^0(x) + \frac{2\big(13xz^2-17xz+3x+z\big)}{xz(1-z)}C_3^0(x) \\
& + \frac{2(xz-2z+1)}{z(1-z)}C_4^0(x) - 4C_2^1(x) + 2C_3^1(x)\bigg) + \mathcal{O}\big(\epsilon^0\big)
\end{aligned}
$$

where

- Hyperlogarithms (Panzer '15 for overview)

$$
\mathrm{H}_{a,\vec{w}}(z) = \int_0^z \frac{\mathrm{d}z'}{z'-a}\,\mathrm{H}_{\vec{w}}(z')
$$

- Integration constants
  - $C_1^0(x), C_2^0(x), C_3^0(x), C_4^0(x), C_2^1(x), C_3^1(x)$ — unknown functions

I.3 Integration Constants for $F(x, z, \epsilon)$

$$F_4^\star(x,\epsilon) = \int_0^1 \mathrm{d}z\, f_4(z)\, F_4(x,z,\epsilon)$$

Calculate RHS

- *direct integration*
  - HyperInt Panzer '14
  - Mellin moments

i	1	2	3	4	5	6	7	8	9	10	11
f_i	1	1	z	$z\,(1-z)$	1	z^2	$1-z$	z	z	z	$z\,(1-z)$

Calculate LHS

- *IBP reduction*
- *differential equations*

$$F_4^\star(x,\epsilon) = \int_0^1 \mathrm{d}z\, z\,(1-z) \int \frac{\mathrm{dPS}(3;\, x, z)}{D_2\, D_6}$$

  - old IBP basis

$$\mathrm{dPS}(3;\, x, z) = x\ q\cdot k_3\ \delta\left(1 - x - (q-k_1)^2\right)\ \delta(x\, z\, q\cdot k_3 - k_1\cdot k_3)\ \mathrm{dPS}(3)$$

  - new IBP basis

$$\mathrm{dPS}(3;\, x) = \int_0^1 \mathrm{d}z\ \mathrm{dPS}(3;\, x, z) = \mathrm{dPS}(3)\,\delta\left(1 - x - (q-k_1)^2\right)$$

II.1 IBP Reduction for $F^\star(x,\epsilon)$

Cutkosky rules for Cut Propagators

$$\delta\big(p_1^2\big)\,\delta\big(p_2^2\big)\,\delta\big(p_3^2\big)\,\delta\big((q-p_1-p_2-p_3)^2\big)\,\delta\big(x-2\,q\cdot p_1\big)\,\cancel{\delta\big(x\,z\,q\cdot p_3-p_1\cdot p_3\big)}$$

Reduction

- do partial fraction decomposition
- identify topologies
- run `LiteRed`
  - 12 masters $G(x,\epsilon)$

$$G_1=\{\} \quad G_2=\{2\} \quad G_3=\{7\} \quad G_4=\{2,7\}$$
$$G_5=\{2,\cancel{6},7\} \quad G_6=\{1,2\} \quad G_7=\{2,3,4,7\} \quad G_8=\{5,7\}$$
$$G_9=\{2,4,5\} \quad G_{10}=\{2,4,5,7\} \quad G_{11}=\{3,5,7\} \quad G_{12}=\{1,4,5,7\}$$

  - denominators

$$D_1=(k_2+k_3)^2 \quad D_2=(q-k_2)^2 \quad D_3=(q-k_1-k_2)^2$$
$$D_4=(q-k_1-k_3)^2 \quad D_5=(q-k_2-k_3)^2 \quad D_6=\big(k_1^2\big)_{\mathrm{x}} \quad D_7=q\cdot k_3$$

II.1 IBP Reduction for $F^\star(x,\epsilon)$

Cutkosky rules for Cut Propagators

$$\delta\left(p_1^2\right)\delta\left(p_2^2\right)\delta\left(p_3^2\right)\delta\left((q-p_1-p_2-p_3)^2\right)\delta\left(x-2\,q\cdot p_1\right)\cancel{\delta\left(x\,z\,q\cdot p_3-p_1\cdot p_3\right)}$$

Reduction

$$\begin{aligned}F_4^\star(x,\epsilon) = {} & \frac{(2-3\epsilon)\left(x+5\epsilon x-2\epsilon^2\,(8-7x)\right)}{4\,\epsilon^2\,x^2\,(4-5x)}\,G_1(x,\epsilon)\\ & + \frac{x\,(1-x)+\epsilon\left(16-33x+15x^2\right)-\epsilon^2\left(48-82x+26x^2\right)}{4\,\epsilon\,x^2\,(4-5x)}\,G_2(x,\epsilon)\\ & + \frac{(1-2\epsilon)\left(x-2\epsilon\,(2-x)\right)}{4\,\epsilon\,x\,(4-5x)}\,G_3(x,\epsilon) - \frac{4-7x+2x^2+\epsilon\,x\,(4-2x)}{4\,x\,(4-5x)}\,G_4(x,\epsilon)\\ & - \frac{3\,x\,(1-x)}{4\,(4-5x)}\,G_5(x,\epsilon)\end{aligned}$$

II.2 Differential Equations for $G(x,\epsilon)$

ODE Definition

$$\frac{\mathrm{d}G(x,\epsilon)}{\mathrm{d}x} = A_G(x,\epsilon)\, G(x,\epsilon)$$

- x as free variable
- IBP rules (step V)
- 12×12 matrix
- alphabet

$$x, \quad x-1, \quad x-2$$

Epsilon form

- `Fuchsia` (Lee algorithm)

$$\frac{\mathrm{d}\hat{G}(x,\epsilon)}{\mathrm{d}x} = \epsilon\, B_G(x)\, \hat{G}(x,\epsilon), \quad \text{where} \quad G(x,\epsilon) = \boxed{T_G(x,\epsilon)}\, \hat{G}(x,\epsilon)$$

Solutions for $\hat{G}(x,\epsilon)$

- fully automated
- any ϵ-order

$$G_5(x,\epsilon) = \frac{2}{x\,\epsilon^2}\Big(30C_1^0 + 6C_2^0 + 6C_3^0 + (14-35x)C_4^0 - 2C_5^0\Big) + \frac{1}{x\,\epsilon}\Big(-390C_1^0 + 60C_1^1 - 78C_2^0$$
$$+ 12C_2^1 - 78C_3^0 + 12C_3^1 - (182-455x)C_4^0 + (28-70x)C_4^1 + 26C_5^0 - 4C_5^1$$
$$+ \Big((60-120x)C_1^0 + (132-144x)C_2^0 - (48-36x)C_3^0 - (112-84x)C_4^0$$
$$+ (16-12x)C_5^0\Big)\mathrm{H}_0(x) + \Big((-480+120x)C_1^0 - (96-144x)C_2^0 - 36(1+x)C_3^0$$
$$+ (-224+336x)C_4^0 + (-8+12x)C_5^0\Big)\mathrm{H}_1(x)\Big) + \mathcal{O}\left(\epsilon^0\right)$$

where

- Hyperlogarithms

$$\mathrm{H}_{a,\vec{w}}(z) = \int_0^z \frac{\mathrm{d}z'}{z'-a}\,\mathrm{H}_{\vec{w}}(z')$$

- Integration constants
  - $C_1^0, C_2^0, C_3^0, C_4^0, C_5^0, C_1^1, C_2^1, C_3^1, C_4^1, C_5^1$ — unknown constants

II.3 Integration Constants for $G(x,\epsilon)$

$$G_5^\star(\epsilon) = \int_0^1 \mathrm{d}x\, g_5(x)\, G_5(x,\epsilon)$$

Calculate RHS

- direct integration
  - `HyperInt` Panzer '14
  - Mellin moments

i	1	2	3	4	5	6	7	8	9	10	11	12
g_i	1	1	1	1	x	1	$(1-x)^2$	1	x	$x(1-x)$	$1-x$	$1-x$

Calculate LHS

- *IBP reduction*
- *direct integration*

$$G_5^\star(\epsilon) = \int_0^1 \mathrm{d}x\, x \int \frac{\mathrm{dPS}(3;\, x)}{D_2\, D_6\, D_7}$$

  - old IBP basis

$$\mathrm{dPS}(3;\, x) = \mathrm{dPS}(3)\, \delta\left(1-x-(q-k_1)^2\right)$$

  - new IBP basis

$$\int_0^1 \mathrm{d}x\, \mathrm{dPS}(3;\, x) = \mathrm{dPS}(3)$$

Cutkosky rules for Cut Propagators

$$\delta\left(p_1^2\right)\delta\left(p_2^2\right)\delta\left(p_3^2\right)\delta\left((q-p_1-p_2-p_3)^2\right)\cancel{\delta\left(x-2q\cdot p_1\right)\delta\left(x\,z\,q\cdot p_3-p_1\cdot p_3\right)}$$

Reduction

- run `LiteRed` Lee '12
  - 2 masters $H(\epsilon)$

$$H_1=\{\}\quad H_2=\{1,2,7\}$$

  - denominators

$$D_1=(k_2+k_3)^2\qquad D_2=(q-k_2)^2\qquad D_3=(q-k_1-k_2)^2$$

$$D_4=(q-k_1-k_3)^2\quad D_5=(q-k_2-k_3)^2\quad D_6=\left(k_1^2\right)_{\mathrm{x}}\qquad D_7=q\cdot k_3$$

- example

$$G_5^\star(\epsilon)=-\frac{2\,(2-3\epsilon)\,(3-4\epsilon)\left(1-7\epsilon+30\epsilon^2-36\epsilon^3\right)}{3\,\epsilon^2\left(1-5\epsilon+6\epsilon^2\right)}\,H_1(\epsilon)$$

III.2 Results for $H(\epsilon)$

Direct Calculation

- known in closed-form

$$H_1(\epsilon) = \frac{1}{12} + \frac{59}{72}\epsilon + \left(\frac{2\,263}{432} - \frac{2}{3}\zeta_2\right)\epsilon^2 + \left(\frac{72\,023}{2\,592} - \frac{59}{9}\zeta_2 - \frac{13}{6}\zeta_3\right)\epsilon^3 + \left(\frac{2\,073\,631}{15\,552} - \frac{2\,263}{54}\zeta_2 - \frac{767}{36}\zeta_3 + \frac{1}{12}\zeta_4\right)\epsilon^4 + \mathcal{O}\left(\epsilon^5\right)$$

- more effortful
- doable to any order (if needed)

$$H_2(\epsilon) = -\frac{4\zeta_3}{\epsilon} - 42\zeta_4 + \mathcal{O}(\epsilon)$$

II.4 Results for $G(x,\epsilon)$

- we know integration constants now
- final result

$$G_5(x,\epsilon) = \frac{1}{3x}\Bigg[-\frac{1}{\epsilon^2} + \frac{\mathrm{H}_0(x) + 4\,\mathrm{H}_1(x)}{\epsilon} - (7-6x)\;\mathrm{H}_{0,0}(x) - 2(5-3x)\;\mathrm{H}_{0,1}(x)$$

$$-2(2+3x)\;\mathrm{H}_{1,0}(x) - 2(5+3x)\;\mathrm{H}_{1,1}(x) - 2(1-3x)\,\zeta_2 + \Bigg(\big(61-54x\big)\,\mathrm{H}_{0,0,0}(x)$$

$$+(46-36x)\,\mathrm{H}_{0,0,1}(x) + 4\,\mathrm{H}_{0,1,0}(x) + 28\,\mathrm{H}_{0,1,1}(x) - 18x\,\mathrm{H}_{0,1,1}(x) + 4\,\mathrm{H}_{1,0,0}(x)$$

$$+18x\,\mathrm{H}_{1,0,0}(x) + 16\,\mathrm{H}_{1,0,1}(x) + 4\,\mathrm{H}_{1,1,0}(x) + 36x\,\mathrm{H}_{1,1,0}(x) + 10\,\mathrm{H}_{1,1,1}(x)$$

$$+54x\,\mathrm{H}_{1,1,1}(x) + \zeta_2\,\big(38\,\mathrm{H}_0(x) - 36x\,\mathrm{H}_0(x) - 16\,\mathrm{H}_1(x)\big) + \big(36-18x\big)\,\zeta_3\Bigg)\,\epsilon\Bigg] + \mathcal{O}\big(\epsilon^2\big)$$

I.4 Results for $F(x,z,\epsilon)$

- we know integration constants now
- final result

$$F_{11}(x,z,\epsilon) = \frac{4}{3(1-x)x^2(1-z)z}\Bigg[\frac{3+x-4xz}{2\epsilon^2} + \frac{1}{\epsilon}\Big(-2(3+x-4xz)H_1(x) - (6-x-5xz)H_0(x) - (3+x-4xz)H_0(z) - (3-x-2xz)H_1(z) + 6(1-xz)H_{1/x}(z)\Big)$$
$$+9(1-x)H_{0,1}(x) + 8(3+x-4xz)H_{1,1}(x) + (24-7x-17xz)H_{0,0}(x) + \Big(2(6-x-5xz)H_0(z) + 4(3-x-2xz)H_1(z) - 15(1-xz)H_{\frac{1}{x}}(z) - 3(1-xz)H_{\frac{1}{x(2-x)}}(z)\Big)H_0(x)$$
$$+H_0(1-x)\Big(5(3+x-4xz)H_0(x) + 4(3+x-4xz)H_0(z) + 4(3-x-2xz)H_1(z) - 21(1-xz)H_{\frac{1}{x}}(z) - 3(1-xz)H_{\frac{1}{x(2-x)}}(z)\Big) + 2(3+x-4xz)H_{0,0}(z) + 2(3-x-2xz)H_{0,1}(z) - 12(1-xz)H_{0,\frac{1}{x}}(z) + 2(3-x-2xz)H_{1,0}(z) + 2(3-2x-xz)H_{1,1}(z)$$
$$-6(2-x-xz)H_{1,\frac{1}{x}}(z) - 9(1-xz)H_{\frac{1}{x},0}(z) - 6(1-xz)H_{\frac{1}{x},1}(z) + 15(1-xz)H_{\frac{1}{x},\frac{1}{x}}(z)$$
$$-3(1-xz)H_{\frac{1}{x(2-x)},0}(z) + 3(1-xz)H_{\frac{1}{x(2-x)},\frac{1}{x}}(z) + 2(3-5x+2xz)\zeta_2\Bigg] + \mathcal{O}(\epsilon)$$

Project `Fuchsia`

- Based on the *Lee algorithm* Lee '14
  - support multivariate systems
  - alternatives: `Canonica`, `epsilon`
- Open-source Gituliar, Magerya '16 '17
  - `http://github.com/gituliar/fuchsia`
- Implemented in `Python`
  - `SageMath`
  - `Maxima`
  - `Maple` (optional)
- Algorithms
  1. **Fuchsification** (Jordan form)
     Get rid of apparent singularities
  2. **Normalization** (eigenvalues, eigenvectors)
     Balance eigenvalues to $\alpha\,\epsilon$ form
  3. **Factorization** (linear equations)
     Reduce to the epsilon form
  4. **Block-triangular optimization**

Summary

Fuchsia project

- full implementation of Lee '14 algorithm
- open-source (`Python`, `SageMath`, `Maxima`)
- `http://github.com/gituliar/fuchsia`
- `arXiv:1607.0079`, `arXiv:1701.04269`

EEC Results

- all virtual terms
- masters for real terms

Cross-check

- Master Integrals
  - higher Mellin moments
- EEC result
  - cancellation of $1/\epsilon^2$ and $1/\epsilon$ poles
  - soft limit for $z \to 0$
  - $N = 4$ SYM Belitsky et al. '13

Thank you!