Higher-Order Corrections in QCD Evolution Equations and Tools for Their Calculation

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Three-loop time-like $q \to g$ splitting function

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- **Why three-loop?**
- **Why time-like?**
- **Why q $\to$ g?**
- **We simply like splitting functions!**

Three-loop time-like $q \to g$ splitting function

Why three-loop?

Because since 1980 two-loop splitting functions were calculated by various methods probably as more times as any other expression.

Space-like

- axial gauge (PV prescription) Curci, Furmanski, Petronzio '80; Ellis, Vogelsang '98
- Feynman gauge Floratos, Kounnas, Lacaze '81
- axial gauge (ML prescription) Bassetto, Heinrich, Kunszt, Vogelsang '98
- Mellin space Moch, Vermaseren '99
- axial gauge (NPV prescription) OG, Jadach, Skrzypek, Kusina '14

Time-like

- axial gauge (PV prescription) Furmanski, Petronzio '80
- Feynman gauge Floratos, Kounnas, Lacaze '81
- analytic continuation Stratmann, Vogelsang '96;
  Blumlein, Ravindran, van Neerven '00; Moch, Vogt '07
- Mellin space Mitov, Moch '06

Three-loop time-like $q \to g$ splitting function

Why time-like?

Because three-loop space-like splitting functions are already calculated. Moch, Vermaseren, Vogt '04

But can't one use some trick to derive them from space-like results?

Examples of tricks: Drell, Levy, Yan '70; Gribov, Lipatov '72

Sure!

- NNLO non-singlet Mitov, Moch, Vogt '06
- NNLO singlet ($q \to q$ and $g \to g$) Moch, Vogt '07
- NNLO singlet ($q \to g$ and $g \to q$) Almasy, Moch, Vogt '11

Three-loop time-like $q \to g$ splitting function

But why $\mathbf{q \to g}$?

**"In summary, these considerations are still not sufficient to definitely fix the right-hand-side of $P_{q\to g}$.
As an estimate of the remaining uncertainty we suggest to use the offset: ...".**

From the paper on NNLO singlet ($q \to g$ and $g \to q$) Almasy, Moch, Vogt '11

Three-loop time-like $\mathbf{q \to g}$ splitting function should be calculated directly.

1. Mass factorization at LO

The unpolarized differential cross-section

$$\frac{1}{\sigma_{\rm tot}}\,\frac{d^2\sigma^H}{dx\,d\cos\theta} = \frac{3}{8}(1+\cos^2\theta)\,\mathcal{F}_T(x) + \frac{3}{4}\sin^2\theta\,\mathcal{F}_L(x) + \frac{3}{4}\cos\theta\,\mathcal{F}_A(x)$$

The mass factorization relations are

Vermaseren, Vogt, Moch '05

$$\begin{aligned}\mathcal{F}_T^{(1)} &= -\frac{2}{\epsilon}\,P_{\rm gq}^{(0)} + c_{T,\rm g}^{(1)} + \epsilon\,a_{T,\rm g}^{(1)} + \epsilon^2 b_{T,\rm g}^{(1)} \\ \mathcal{F}_{L,\rm g}^{(1)} &= c_{L,\rm g}^{(1)} + \epsilon\,a_{L,\rm g}^{(1)} + \epsilon^2 b_{L,\rm g}^{(1)}\end{aligned}$$

Kinematic variables:

$$x = \frac{2k_0 q}{q^2} \qquad q^2 = s > 0 \qquad 0 < x \le 1$$

1. Mass factorization at NLO

The unpolarized differential cross-section

$$\frac{1}{\sigma_{\rm tot}}\,\frac{d^2\sigma^H}{dx\,d\cos\theta} = \frac{3}{8}(1+\cos^2\theta)\,\mathcal{F}_T(x) + \frac{3}{4}\sin^2\theta\,\mathcal{F}_L(x) + \frac{3}{4}\cos\theta\,\mathcal{F}_A(x)$$

The mass factorization relations are

Vermaseren, Vogt, Moch '05

$$\begin{aligned}
\mathcal{F}^{(2)}_{T,\rm g} \;&=\; \frac{1}{\epsilon^2}\left\{P^{(0)}_{\rm gq}\left(P^{(0)}_{\rm qq}+P^{(0)}_{\rm gg}+\beta_0\right)\right\} + \frac{1}{\epsilon}\left\{P^{(1)}_{\rm gq} + 2\,c^{(1)}_{T,\rm q}\,P^{(0)}_{\rm gq} + c^{(1)}_{T,\rm g}\,P^{(0)}_{\rm gg}\right\} \\
&\quad + c^{(2)}_{T,\rm g} - 2\,a^{(1)}_{T,\rm q}\,P^{(0)}_{\rm gq} - a^{(1)}_{T,\rm g}\,P^{(0)}_{\rm gg} + \epsilon\left\{a^{(2)}_{T,\rm g} - 2\,b^{(1)}_{T,\rm q}\,P^{(0)}_{\rm gq} - b^{(1)}_{T,\rm g}\,P^{(0)}_{\rm gg}\right\} \\
\mathcal{F}^{(2)}_{L,\rm g} \;&=\; \frac{1}{\epsilon}\left\{2\,c^{(1)}_{L,\rm q}\,P^{(0)}_{\rm gq} + c^{(1)}_{L,\rm g}\,P^{(0)}_{\rm gg}\right\} + c^{(2)}_{L,\rm g} - 2\,a^{(1)}_{L,\rm q}\,P^{(0)}_{\rm gq} - a^{(1)}_{L,\rm g}\,P^{(0)}_{\rm gg} \\
&\quad + \epsilon\left\{a^{(2)}_{L,\rm g} - 2\,b^{(1)}_{L,\rm q}\,P^{(0)}_{\rm gq} - b^{(1)}_{L,\rm g}\,P^{(0)}_{\rm gg}\right\}
\end{aligned}$$

1. Mass factorization at NNLO

The unpolarized differential cross-section

$$\frac{1}{\sigma_{\rm tot}}\frac{d^2\sigma^H}{dx\, d\cos\theta} = \frac{3}{8}(1+\cos^2\theta)\,\mathcal{F}_T(x) + \frac{3}{4}\sin^2\theta\,\mathcal{F}_L(x) + \frac{3}{4}\cos\theta\,\mathcal{F}_A(x)$$

The mass factorization relations are

Moch, Vogt '07

$$\begin{aligned}
\mathcal{F}^{(3)}_{T,\rm g} \;=\; & -\frac{1}{6\epsilon^3}\Big\{P^{(0)}_{\rm gi}P^{(0)}_{\rm ij}P^{(0)}_{\rm jg} + 3\beta_0\,P^{(0)}_{\rm gi}P^{(0)}_{\rm ig} + 2\,\beta_0^2\,P^{(0)}_{\rm gg}\Big\} \\
& + \frac{1}{6\epsilon^2}\Big\{2P^{(0)}_{\rm gi}P^{(1)}_{\rm ig} + P^{(1)}_{\rm gi}P^{(0)}_{\rm ig} + 2\beta_0\,P^{(1)}_{\rm gg} + 2\beta_1\,P^{(0)}_{\rm gg} + 3P^{(0)}_{\rm gi}\Big(P^{(0)}_{\rm ij} + \beta_0\delta_{\rm ij}\Big)\,c^{(1)}_{\phi,\rm j} \\
& - \frac{1}{6\epsilon}\Big\{2\,P^{(2)}_{\rm gg} + 3\,P^{(1)}_{\rm gi}c^{(1)}_{\phi,\rm i} + 6\,P^{(0)}_{\rm gi}c^{(2)}_{\phi,\rm i} - 3P^{(0)}_{\rm gi}\Big(P^{(0)}_{\rm ij} + \beta_0\delta_{\rm ij}\Big)\,a^{(1)}_{\phi,\rm j}\Big\} \\
& + c^{(3)}_{\phi,\rm g} - \frac{1}{2}\,P^{(1)}_{\rm gi}a^{(1)}_{\phi,\rm i} - P^{(0)}_{\rm gi}a^{(2)}_{\phi,\rm i} + \frac{1}{2}\,P^{(0)}_{\rm gi}\Big(P^{(0)}_{\rm ij} + \beta_0\delta_{\rm ij}\Big)\,b^{(1)}_{\phi,\rm j} + \ldots \\
\mathcal{F}^{(3)}_{L,\rm g} \;=\; & \ldots
\end{aligned}$$

2. Fragmentation Functions

The hadronic tensor is defined as

$$W_{\mu\nu}(x,\epsilon) = \frac{x^{d-3}}{4\pi} \int \mathrm{dPS(n)}\, M_\mu(n)\, M_\nu(n)$$

where $\mathrm{dPS(n)}$ is n-particle phase-space and amplitude $M^\mu(n)$ describes process

$$\gamma^*(q) \to g(k_0) + q(k_1) + \bar{q}(k_2) + (\text{other } n-2 \text{ partons})$$

Fragmentation functions are defined as

$$\begin{aligned}
\mathcal{F}_T(x,\epsilon) &= \frac{2}{2-d}\left(\frac{k_0\cdot q}{q^2} W_\mu{}^\mu + \frac{k_0^\mu k_0^\nu}{k_0\cdot q} W_{\mu\nu}\right) \\
\mathcal{F}_L(x,\epsilon) &= \frac{k_0^\mu k_0^\nu}{k_0\cdot q} W_{\mu\nu} \\
\mathcal{F}_A(x,\epsilon) &= -i\,\frac{2}{(d-2)(d-3)}\frac{k_0^\alpha q^\beta}{q^2}\,\epsilon_{\mu\nu\alpha\beta} W_{\mu\nu}
\end{aligned}$$

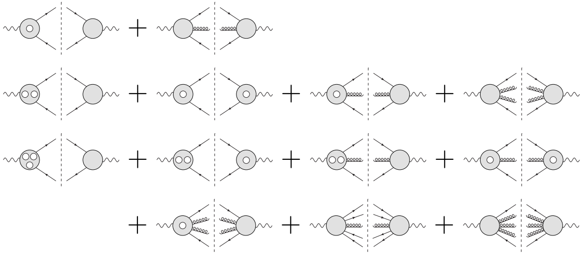
2. Fragmentation Functions

$$\mathcal{F}_T(x,\epsilon) = \frac{2}{2-d}\left(\frac{p\cdot q}{q^2} W_{\mu}{}^{\mu} + \frac{p^{\mu}p^{\nu}}{p\cdot q} W_{\mu\nu}\right)$$

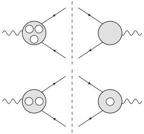
$\mathcal{F}_T^{(1)} =$

$\mathcal{F}_T^{(2)} =$

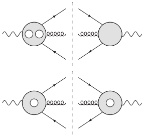
$\mathcal{F}_T^{(3)} =$



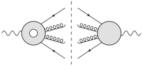
3. Final Considerations



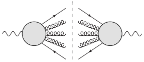
Pure-virtual contributions contain overall $\delta(1-x)$ factor.
We do not consider such contributions.



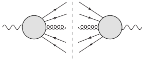
Can be extracted from Garland, Gehrmann, Glover, Koukoutsakis, Remiddi '01
Calculated by Duhr, Gehrman, Jaquier 1411.3587 [hep-ph]



One-loop helicity amplitudes by Bern, Dixon, Kosower '97
Final-state integration is of NLO complexity — simple.

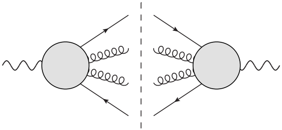


Contribution is known from analytical continuation by Almasy, Moch, Vogt '11

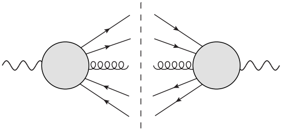


Unknown!

Final-state integration



$\sim$ 3-loop



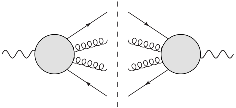
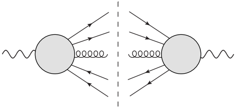
$\sim$ 4-loop

The main challenge of the calculation is *n*-particle final-state integration:

$$\int \mathrm{dPS(n)} = \int \prod_{i=0}^{n-1} \mathrm{d}^m k_i \delta^+(k_i^2)\ \delta\left(x - \frac{2k_0 \cdot q}{q^2}\right)\ \delta(q - \sum_{j=0}^{n-1} k_j)$$

We attack such integrals with IBP identities and differential equations.

Preparation

		
QGRAF	▶ 8 amplitudes	▶ 48 amplitudes
FORM		
▶ trace of gamma matrices		
▶ index contraction		
▶ color traces		
▶ partial fractioning		
Mathematica	▶ 499 integrals	▶ 55 614 integrals
▶ analyze symmetries		
▶ split by topologies		
LiteRed		
▶ find IBP reduction rules	▶ ∼10 h	▶ ∼350 h (10 threads)
▶ find masters	▶ 9 masters	▶ ∼76 masters

System of Differential Equations for Masters at NLO

$$
\begin{pmatrix}
\frac{(2\epsilon-1)(2x-1)}{(1-x)x} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{3\epsilon-2}{(1-x)x} & -\frac{3\epsilon-1}{x} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{(2\epsilon-1)(3\epsilon-1)}{\epsilon(x-1)x} & \frac{2\epsilon}{1-x} & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{(2\epsilon-1)(3\epsilon-2)(3\epsilon-1)}{2\epsilon(x-1)x^2} & \frac{(2\epsilon-1)(3\epsilon-1)\left(x^2-10x+1\right)}{2(1-x)x^2(x+1)} & 0 & \frac{2\epsilon\left(x^2-3x-2\right)}{(1-x)x(x+1)} & \frac{2\epsilon(6\epsilon-1)}{(1-x)x} & 0 & 0 & 0 & 0 \\
0 & \frac{(2\epsilon-1)(3\epsilon-1)}{\epsilon(x-1)x} & 0 & \frac{2}{x-1} & \frac{6\epsilon-1}{1-x} & 0 & 0 & 0 & 0 \\
\frac{4(2\epsilon-1)(3\epsilon-2)(3\epsilon-1)}{\epsilon^2(1-x)x^3(x+1)} & \frac{4(2\epsilon-1)(3\epsilon-1)\left(x^2-x+1\right)}{\epsilon(x-1)x^3(x+1)^2} & 0 & \frac{4\left(x^2+1\right)}{(x-1)x^2(x+1)^2} & \frac{2(6\epsilon-1)}{(1-x)x^2(x+1)} & \frac{(2\epsilon+1)(2x+1)}{-x(x+1)} & 0 & 0 & 0 \\
\frac{2(2\epsilon-1)(3\epsilon-2)(3\epsilon-1)}{\epsilon^2(x-1)^2x^2} & -\frac{(2\epsilon-1)(3\epsilon-1)}{\epsilon(x-1)x^2} & \frac{2\epsilon}{(x-1)x} & 0 & 0 & 0 & \frac{4\epsilon+1}{-x} & 0 & 0 \\
\frac{2(1-2\epsilon)(3\epsilon-2)(3\epsilon-1)}{\epsilon^2(x-1)^2x^3} & \frac{2(2\epsilon-1)(3\epsilon-1)(3x-1)}{\epsilon(x-1)^3x^3} & \frac{4\epsilon}{(1-x)^2x} & \frac{4\left(x^2+1\right)}{(x-1)^3x^2} & \frac{2(6\epsilon-1)(x+1)}{(1-x)^3x^2} & 0 & 0 & \frac{(2\epsilon+1)(2x-1)}{(1-x)x} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{(2\epsilon+1)(2x-1)}{(1-x)x}
\end{pmatrix}
$$

- Alphabet (letters): $\{x, 1-x, 1+x\}$
- Singular in $\epsilon \to 0$ limit (can be fixed)
- Non-triangular terms are $\sim \epsilon$
- Main diagonal contains letters in -1st power

Similar properties are observed by Gehrmann, von Manteuffel, Tancredi, Weihs '14 for two-loop masters in $q\bar{q} \to V\bar{V}$ with massive bosons.

System of Differential Equations for Masters at NLO

It seems possible to solve an arbitrary system of DEs with:

- Alphabet (letters): $\{x, 1-x, 1+x\}$
  $\rightarrow$ solution in terms of HPLs
- Non-triangular terms are $\sim \epsilon$
  $\rightarrow$ allows to apply Henn's method Henn '13
- Main diagonal contains letters in -1st power
  $\rightarrow$ solution in terms of HPLs

To be continued...

Summary

Done:

- **Optimizing** input for `LiteRed`
  $\rightarrow$ reduces 55 614 integrals to just $\sim$ 80 masters
- **IBP identities** for NNLO case
  $\rightarrow$ needs another $\sim$ 200 hours of CPU time
- **General algorithm** to solve differential equations with particular properties

In progress:

- Implementation of the differential equations solver
- Thinking on the boundary conditions finder
- Three-loop time-like $q \rightarrow g$ splitting function